Homothetic Function

A function $f(x_1, x_2 \ldots x_n)$ is said to be homothethic if it satisfieds the following condition.

| vector
| $x = (x_1, x_2 \ldots x_n)$
| $y = (y_1, y_2 \ldots y_n)$

* x and Y $\in$ K and $f(x) = f(y)$
  x and y are not variable, these are vector. and t is some positive constant, if function is homothetic then $f(x) = f(y) \Rightarrow f(tx) = f(ty)$

1.3 Homothetic Functions

Definition 3 *A function $\nu : R^n \to R$ is called homothetic if it is a monotonic transformation of a homogenous function, that is there exist a strictly increasing function $g : R \to R$ and a homogenous function $u : R^n \to R$ such that $\nu = g \circ u$.*

It is clear that homothetiticy is ordinal property: monotonic transformation of homothetic function is homothetic (prove it!).
Examples. Let $u(x, y) = xy$, a homogenous function of degree 2. Then the monotonic transformations

$$g_1(z) = z + 1,\ g_2(z) = z^2 + z,\ g_3(z) = ln\ z$$

generate the following homothetic (but not homogenous) functions

$$\nu_1(x, y) = xy + 1,\ \nu_2(x, y) = x^2y^2 + xy,\ \nu_3(x, y) = ln\ x + ln\ y.$$

1.3.1 Properties of Homothetic Functions

Theorem 7 *Level sets of a homothetic function are radial expansions of one another, that is $\nu(x) = \nu(y)$ implies $\nu(tx) = \nu(ty)$ for arbitrary $x = (x_1, ..., x_n),\ y = (y_1, ..., y_n),\ t > 0$.*

Proof. $\nu(tx) = g(u(tx)) = g(t^k u(x)) = g(t^k u(y)) = g(u(ty)) = \nu(ty)$.

Theorem 8 *For a homothetic function the slopes of level sets along rays from the origin are constant, that is*

$$-\frac{\frac{\partial \nu}{\partial x_i}(tx)}{\frac{\partial \nu}{\partial x_j}(tx)} = -\frac{\frac{\partial \nu}{\partial x_i}(x)}{\frac{\partial \nu}{\partial x_j}(x)}$$

for all i, j and $t > 0$.

In other words Marginal Rate of Substitution (MRS) for a homothetic function is a homogenous function of degree 0.
Proof.

$$\frac{\frac{\partial \nu}{\partial x_i}(tx)}{\frac{\partial \nu}{\partial x_j}(tx)} = \frac{\frac{\partial}{\partial x_i}g(u(tx))}{\frac{\partial}{\partial x_j}g(u(tx))} = \frac{g'(u(tx))\cdot\frac{\partial u}{\partial x_i}(tx)}{g'(u(tx))\cdot\frac{\partial u}{\partial x_j}(tx)} = \frac{\frac{\partial u}{\partial x_i}(tx)}{\frac{\partial u}{\partial x_j}(tx)} = \frac{t^{k-1}\frac{\partial u}{\partial x_i}(x)}{t^{k-1}\frac{\partial u}{\partial x_j}(x)} = \frac{\frac{\partial u}{\partial x_i}(x)}{\frac{\partial u}{\partial x_j}(x)}.$$

Substituting $t = 1$ we obtain

$$\frac{\frac{\partial \nu}{\partial x_i}(x)}{\frac{\partial \nu}{\partial x_j}(x)} = \frac{\frac{\partial u}{\partial x_i}(x)}{\frac{\partial u}{\partial x_j}(x)},$$

Thus

$$-\frac{\frac{\partial \nu}{\partial x_i}(tx)}{\frac{\partial \nu}{\partial x_j}(tx)} = -\frac{\frac{\partial u}{\partial x_i}(x)}{\frac{\partial u}{\partial x_j}(x)} = -\frac{\frac{\partial \nu}{\partial x_i}(x)}{\frac{\partial \nu}{\partial x_j}(x)},$$

this completes the proof.

20.13 Homothetic Functions

Homothetic functions are the ordinal equivalent of homogeneous functions.

Homothetic Function. Let C be a cone. A function $f : C \to \mathbb{R}$ is *homothetic* if for every $\mathbf{x}, \mathbf{y} \in C$ and $t > 0$, $f(\mathbf{x}) \geq f(\mathbf{y})$ if and only if $f(t\mathbf{x}) \geq f(t\mathbf{y})$.

One consequence of the definition of homotheticity is that f is equivalent to g defined by $g(\mathbf{x}) = f(t\mathbf{x})$.

Any homogeneous utility function is also homothetic. And since homotheticity is an ordinal property, any increasing transformation of a homogeneous utility function is homothetic too.

However, not all homothetic preferences have a homogeneous utility representation. Lexicographic preferences are homothetic, but cannot be represented by any utility function—homogeneous or otherwise. If we require that preferences are also monotonic and continuous, we can represent homothetic preferences by a homogeneous utility function.

For a utility function, homotheticity means that preferences are invariant under scalar multiplication in the sense that the set of indifference curves is unchanged when all consumption bundles are multiplied by the same positive number. More precisely, preferences are invariant under homothetic transformations centered on the origin.

Homothetic preferences include commonly used functional forms such as Cobb-Douglas utility and constant elasticity of substitution utility.

One special type of homothetic utility is homogeneous utility, where multiplying the consumption bundle by a scalar multiplies utility by some power of that scalar. The Homothetic Representation Theorem will show that monotonic continuous and homothetic preferences can be represented by a homogeneous utility function.

Homotheticity in economics is based on comparing positive scalar multiples of vectors. By restricting our attention to consumption sets that are cones, we ensure that scalar multiplication is always possible. Such scaling preserves the shapes of objects, including indifference surfaces. It only changes their scale.